Graphs and Their Applications (9)

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26 Bipartite graphs

In our previous article [2], we discussed the following assignment problem.

Five applicants A, B, C, D, E apply to work in a company. There are six jobs available: $J_1, \cdots, J_6$. *Applicant A is qualified for jobs* J_2 *and* J_6; *B is qualified for jobs* J_1, J_3 *and* J_4; *C is qualified for jobs* J_2, J_3 *and* J_6; *D is qualified for jobs* J_1, J_2 *and* J_3; *E is qualified for all jobs except* J_4 *and* J_6. *Is it possible to assign each applicant to a job for which he/she is qualified?*

We then solved the problem with the use of a bipartite graph (see Figure 26.1). The bold edges indicate the assignment of the job to the respective applicant.

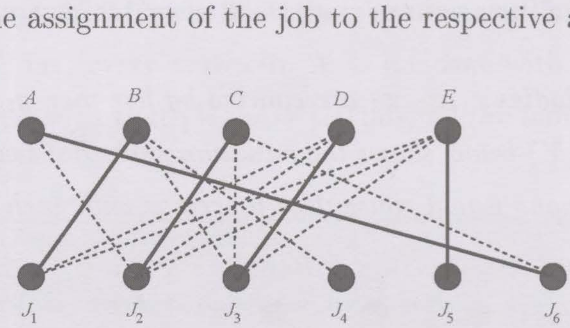

Figure 26.1

Such a need for assignments or matchings of two groups occurs frequently in real life. In a small community, the eligible men may need to be matchmade to the eligible ladies.

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In a badminton tournament, players in the same team may need to be paired to form doubles. And in the old days, riders may need to be assigned to horses in an army. In each of these situations, the following are true:

(i) Each entity may be matched to at most one other entity. For example, each eligible man can only be matched to one lady.

(ii) Some entities may not be matched to some other entities. For example, a particular man may not want to marry a particular lady.

In such situations, we may use a graph to help us find a suitable assignment. We may represent a situation by a graph G where a vertex denotes an entity and two vertices are adjacent if and only if the two entities can *possibly* be matched. In our example at the beginning, a suitable assignment is represented by a subgraph of G and we notice that the degree of any vertex in this subgraph is at most one. This motivates us to introduce the following:

> Let G be a graph. A nonempty set M of edges in G is called a **matching** of G if no two edges of M are incident with a common vertex.

Observe that in our three situations (the marriage matchmaking, the badminton pairs, and the horse and rider), the first and last can be represented by bipartite graphs. For example, in the marriage matchmaking situation, the men and the ladies form two partite sets. In this article, we shall restrict our discussion of matching to bipartite graphs.

Exercise 26.1 *Three ladies x_1, x_2, x_3 are courted by five men $y_1, \cdots, y_5$. The bipartite graph with bipartition (X, Y) below shows the situation such that two vertices are adjacent if and only if the two persons would consent to marriage with each other.*

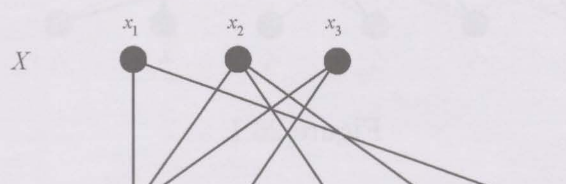

Figure 26.2

(i) Is $\{x_1y_1\}$ a matching?

(ii) Is $\{x_1y_1, x_2y_1\}$ a matching?

(iii) Is $\{x_1y_5, x_2y_1, x_3y_2\}$ a matching?

(iv) Find the number of matchings with three edges.

Exercise 26.2 *Four riders r_1, r_2, r_3, r_4 are to be assigned to four horses h_1, h_2, h_3, h_4. The bipartite graph G with bipartition (R, H) below shows the situation such that two vertices are adjacent if and only if the horse and the rider can get along.*

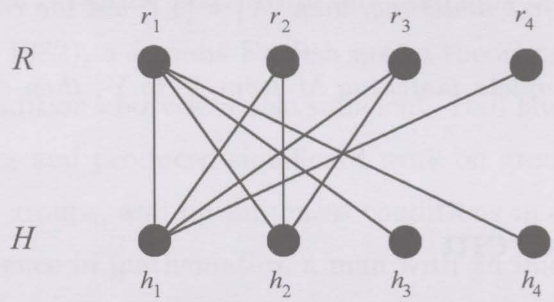

Figure 26.3

(i) Find in G a matching with three edges.

(ii) Does G contain a matching with four edges? Why?

In the bipartite graph G of Figure 26.2, there exists a matching $M = \{x_1y_1, x_2y_3, x_3y_2\}$ such that $|M| = |X|$, i.e., every vertex in X is incident with an edge in M. However, there is no matching M such that $|M| = |Y|$. Thus, all the ladies can be married but not all the men.

> Let G be a bipartite graph with bipartition (X, Y). A matching M in G is called a **complete matching from X to Y** if $|M| = |X|$ i.e., every vertex in X is incident with an edge in M.

We have seen that it is possible that a bipartite graph G with bipartition (X, Y) may have a complete matching from X to Y, but not have a complete matching the other way round. In Exercise 26.2, we have a graph G that does not have a complete matching from any partite set to the other. On the other hand, it is possible for a bipartite graph with bipartition (X, Y) to have a complete matching from X to Y and vice-versa. Such a matching is called perfect.

Let G be graph. A matching M in G is said to be **perfect** if every vertex in G is incident with an edge in M. In particular, if G is bipartite with bipartition (X, Y), then M is **perfect** if $|X| = |M| = |Y|$.

Exercise 26.3 *Prove the following statements for a bipartite graph G with bipartition (X, Y).*

(i) If G contains a complete matching from X to Y, then $|X| \leq |Y|$; but the converse is not true.

(ii) If G contains a perfect matching, then $|X| = |Y|$; but the converse is not true.

(iii) If G contains a complete matching M from X to Y, then M is perfect if and only if $|X| = |Y|$.

27 Hall's Theorem

The first two examples of situations where we need to find a matching lead us to the following two well-known problems:

The Assignment Problem. There are m applicants and n jobs and each applicant is applying for a number of these jobs. Under what conditions is it possible to assign each applicant to a job for which he or she is applying?

The Marriage Problem. There are m men and n women and each man is acquainted with a certain number of the women. Under what conditions is it possible to marry off these m men in such a way that each man marries a woman he is acquainted with?

Following from our discussion on matchings in bipartite graphs, these problems can be reformulated using graph terminology as follows:

Problem. Let G be a bipartite graph with bipartition (X, Y). Under what conditions is there a complete matching from X to Y?

Before we begin to solve the problem, we have the following definition.

Definition 27.1 *Let G be a graph with vertex set V. For a vertex $v \in V$, the* **neighbourhood** *of v, denoted by $N(v)$, is the set of vertices in V that are adjacent to v in G. For a set $S \subseteq V$, the neighbourhood of S, denoted by $N(S)$, is the set of vertices in V that are adjacent to some $v \in S$ in G.*

We shall explore the problem by considering an obvious condition: the number of elements in any subset S of X is at most as many as the number of elements in the neighbourhood $N(S)$ of S.

That this is a necessary condition for G to have a complete matching from X to Y follows quite clearly from the fact that if there is a subset S such that $|S| > |N(S)|$, then not all of the elements of S can be matched in any matching of G.

Philip Hall (1904 - 1982), a famous English group theorist, proved in 1935 that the obviously necessary condition above was also sufficient. Hall had a considerable influence on English Mathematics and produced significant work on groups of prime power order, on p-lengths of soluble groups, and on finiteness conditions in soluble groups. Hall was, in addition to his eminence in mathematics, a man with an immensely broad knowledge ranging from agriculture to music.

Philip Hall (1904 - 1982)

Before stating Hall's Theorem and proving it formally, we need to recall a definition from [1].

Definition 27.2 *Let G be a graph with vertex set V and edge set E. For a set $S \subseteq V$, the* **subgraph induced by** *S, denoted by $[S]$, is the graph H with vertex set S and edge set F where $uv \in F$ if and only if $uv \in E$ and $u, v \in V$.*

We shall now state Hall's Theorem and prove it.

Theorem 27.1 *Let G be a bipartite graph with bipartition (X, Y). Then G contains a complete matching from X to Y if and only if $|S| \leq |N(S)|$ for every subset S of X.*

We shall now give a proof of the sufficiency.

Proof. Assume that in G we have

(*) $\qquad |S| \leq |N(S)|$ for every subset S of X.

We shall show that G contains a complete matching from X to Y by induction on $|X|$.

The statement is obviously true when $|X| = 1$. Assume that the statement is true for $|X| < k$, i.e., G contains a complete matching from X to Y when $|X| < k$, where $k \geq 2$.

Now suppose that $|X| = k$. We consider two cases.

Case (1). $|S| \leq |N(S)| - 1$ for every nonempty proper subset S of X.
Let $x \in X$. Clearly there exists $y \in Y$ such that $xy \in E(G)$. Let $G' = G - \{x, y\}$.

Consider $S \subset X - \{x\}$. Let N be the neighbourhood of S in G. Thus $|S| \leq |N| - 1$ in G. If $y \in N$, then $|S| \leq |N \setminus \{y\}|$ in G' and so $|S| \leq |N(S)|$ in G'. If $y \notin N$, then $|S| \leq |N| - 1$ in G and thus $|S| \leq |N(S)| - 1$ in G'. Hence, G' satisfies (*). By the induction hypothesis, G' contains a complete matching M' from $X \setminus \{x\}$ to $Y \setminus \{y\}$. It follows that $M' \cup \{xy\}$ is a complete matching from X to Y.

Case (2). $|R| = |N(R)|$ for some nonempty proper subset R of X.
Consider the subgraph $G' = [R \cup N(R)]$, i.e., the subgraph induced by $R \cup N(R)$. Clearly, G' is bipartite and satisfies (*). Since $|R| < |X|$, by the induction hypothesis, G' contains a complete matching M' from R to $N(R)$.

Next, consider $G'' = G - (R \cup N(R))$. Suppose there exists $S \subseteq X \setminus R$ such that $|S| > |N(S)|$ in G''. Then $|R \cup S| > |N(R \cup S)|$ in G, which is a contradiction. Thus, $|S| \leq |N(S)|$ in G'' for all $S \subseteq X \setminus R$ and so G'' satisfies (*). By the induction hypothesis, G'' contains a complete matching M'' from $X \setminus R$ to $Y \setminus N(R)$.

Hence $M' \cup M''$ is a complete matching from X to Y in G. □

28 Applications of Hall's Theorem

To show that a given bipartite graph G with bipartition (X, Y) has a complete matching from X to Y by checking that $|S| \leq |N(S)|$ for all subsets S of X is, of course, a tough job. There is, however, a class of bipartite graphs for which the checking is much easier.

Theorem 28.1 *Let G be a bipartite graph with bipartition (X, Y). If there exists a positive integer k such that $d(y) \leq k \leq d(x)$ for all x in X and y in Y, then G has a complete matching from X to Y.*

Proof. We shall apply Hall's theorem to prove the result. Let $S \subseteq X$. Our aim is to show that $|S| \leq |N(S)|$. Let p be the number of edges in G incident with a vertex in S and q the number of edges in G incident with a vertex in $N(S)$. By definition of $N(S)$, every edge incident with a vertex in S is also incident with a vertex in $N(S)$. Thus, $p \leq q$. By assumption, we have

$$p = \sum_{x \in S} d(x) \geq k|S|$$

and

$$q = \sum_{y \in |N(S)|} d(y) \leq k|N(S)|.$$

Thus, we have

$$k|S| \leq p \leq q \leq k|N(S)|,$$

and so $|S| \leq |N(S)|$, as was to be shown. □

In particular, we have the following result.

Corollary 28.1 *Every k-regular bipartite graph, where $k \geq 1$, has a perfect matching*

Proof. Let G be a k-regular bipartite graph with bipartition (X, Y). By Theorem 28.1, G has a complete matching M from X into Y. By assumption,

$$k|X| = \sum_{x \in X} d(x) = e(G) = \sum_{y \in Y} d(y) = k|Y|$$

(see Proposition 23.1[2]). Thus, $|X| = |Y|$, and so M is a perfect matching. □

Exercise 28.1 *Is the converse of Theorem 28.1 true? Justify your answer.*

Exercise 28.2 *There are* 8 *clubs in a school. Each club has at least* 5 *students as its members, and no student is a member of more than* 4 *clubs. Is it possible to form a committee of* 8 *students such that each club has a member in the committee?*

Exercise 28.3 *Let G be a bipartite graph with bipartition (X, Y). Assume that $d(x) \geq 4$ and $d(y) \leq 6$ for all x in X and y in Y. Show that G contains a matching M such that $|M| \geq \frac{2}{3}|X|$.*

In the life sciences, protein structures can be compared using bipartite graph matching [4]. An efficient Multiple Object Vision Tracking System using bipartite graph matching was proposed by Rowan and Maire [3]. Of interest to us also is a variant of the marriage problem. Here, the matchmaker wants a matching of the men and the ladies such that no two persons who rate each other highly are each matched to someone else lower in their estimation. In this variant, each of the men rank each of the ladies in terms of desirability, and vice-versa. We want to find a matching with the property that there are no marriages of the form (x_1, y_1) and (x_2, y_2), but actually x_1 and y_2 prefer each other to their own spouses. In real life, these two may eventually divorce their spouses to marry each other. A matching without any such couples is said to be stable. Remarkably, no matter how the men and ladies rate each other, there is always at least one stable matching! In the United States, assigning of medical residents to hospitals makes use of this fact to obtain a stable matching.

References

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